25/05/2020

# Compactness

(1) Cover :- Let $(X, \mathfrak{I})$ be a topological space and $A \subset X$. Let $G$ denote a family of subsets of $X$. $G$ is called a cover of $A$ if $A \subset \cup\{G : G \in \mathcal{G}\}$

(i) If every member of $G$ is an open set, then the cover $\mathcal{G}$ is called an open cover.

(ii) If $\exists\ \mathcal{G}_1 \subset \mathcal{G}$ s.t. $\mathcal{G}_1$ is a finite set and that $\{G : G \in \mathcal{G}_1\}$ is a cover of $A$ then $\mathcal{G}_1$ is called a finite subcover of the original cover.

(2) Reducible :- Let $(X, \mathfrak{I})$ be a topological space and $A \subset X$. If $\exists$ finite subcover of a cover $\mathcal{G}$ of $A$. Then the cover $\mathcal{G}$ of $A$ is said to be reducible.

(3) Compact Set :- Let $(X, \mathfrak{I})$ be a topological space and $A \subset X$. $A$ is said to be a compact set if every open covering of $A$ is reducible to finite subcovering.

Ex :- If $\mathfrak{I} = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$ and $X = \{a,b,c\}$,
(i) $C_1 = \{\{a\}, \{b,c\}\}$, (ii) $C_2 = \{\{a\}, \{b\}, \{c\}\}$, (iii) $C_3 = \{X\}$

Sol :- (i) $C_1$ is not $\mathfrak{I}$-open cover for $X$, because $\{b,c\} \notin \mathfrak{I}$

(ii) $C_2$ is not $\mathfrak{I}$-open cover for $X$ as $\{c\} \notin \mathfrak{I}$

(iii) $C_3$ is $\mathfrak{I}$-open cover for $X$

M.S. Ex :- Show that $G_n = \left\{\left(0, \frac{n}{n+1}\right) : n \in N\right\}$ is a cover of $(0,1)$.

Sol :- Here $X = (0,1)$ and $G_n = \left(0, \frac{n}{n+1}\right)\ \forall\ n \in N$

Then $G_1 = \left(0, \frac{1}{2}\right)$, $G_2 = \left(0, \frac{2}{3}\right)$, $G_3 = \left(0, \frac{3}{4}\right)$

$G_m = (0,1)$

Now $X = (0,1) = G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_m$

Hence $C$ is cover for $X$

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Locally Compact :- Let $(X, \mathfrak{J})$ be a topological Space and let $x \in X$ be arbitrary. Then X is to be locally compact at x if the closure of any nbd of x is compact.

~~Problem :- Show by means of an example that locally compact need not be compact.~~

~~Sol~~ :-

M3 Thm: A closed subset of a compact space is compact

Proof Let $(X, \mathfrak{J})$ be a compact space and $A \subset$ be closed.

To Prove that A is compact.

Let $\{G_i\}$ be an open Covering of A then

$$A \subset \bigcup_i G_i$$

Here X-A is open

therefore $(X-A) \cup (\cup G_i) = X$

This shows that family consisting of the set X-A and $G_i$'s is an open Covering of X which is known to be compact.

Hence the covering must be reducible to a finite sub-covering

therefore $X-A, G_{i_1}, G_{i_2}, \text{---}, G_{i_n}$

Then $X = (X-A) \cup \left( \bigcup_{r=1}^{n} G_{i_r} \right)$

we claim $A \subset \bigcup_{r=1}^{n} G_{i_r}$, suppose not

$\exists$ $a \in A$ $\ni$ $a \notin \bigcup_{r=1}^{n} G_{i_r}$, also $a \notin X-A$

means a does not belongs $X-A, G_{i_1}, \text{---} G_{i_n}$

which is a Contradiction

$\Rightarrow$ A is compact